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Existence of Positive Solutions for m-Laplacian Boundary Value Problems

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Abstract—Sufficient conditions for the existence of positive solutions of the nonlinear m-Laplacian boundary value problem

$$\left.\begin{aligned} \left(|u'(t)|^{m-2}u'(t)\right)' + f(t,u(t)) = 0, \qquad 0<t<1, \qquad &\text{(E)} \\ u'(0) = u(1) = 0, \qquad &\text{(BC)} \end{aligned}\right\} \qquad \text{(BVP)}$$

are constructed, where $m \geq 2$ and $f : [0,1) \times (0,\infty) \to (0,\infty)$ satisfying $f(t,u)$ is locally Lipschitz continuous for $u \in (0,\infty)$, and $f(t,u)/u^{m-1}$ is strictly decreasing in $u > 0$ for each fixed $t \in (0,1)$.
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Keywords—m-Laplacian boundary value problems, Positive solution, Locally Lipschitz continuous.

1. INTRODUCTION

Many authors have considered the uniqueness and existence of positive solutions for the m-Laplacian boundary value problems

$$\left.\begin{aligned} \left(|u'(t)|^{m-2}u'(t)\right)' + f(t,u(t)) = 0, \qquad 0<t<1, \qquad &\text{(E)} \\ u'(0) = u(1) = 0, \qquad &\text{(BC)} \end{aligned}\right\} \qquad \text{(BVP)}$$

where $m \geq 2$ and $f : [0,1) \times (0,\infty) \to (0,\infty)$ satisfying $f(t,u)$ is locally Lipschitz continuous for $u \in (0,\infty)$, and $f(t,u)/u^{m-1}$ is strictly decreasing in $u > 0$ for each fixed $t \in (0,1)$.

Equations of the type (E) arise in studies of radially symmetric solutions (i.e., solutions u that depend only on the variable $r = |x|$) of the m-Laplacian equation,

$$\nabla \cdot \left(|\nabla u|^{m-2}\nabla u\right) + g(|x|,u) = 0, \qquad R_0 < |x| < R_1, \quad x \in \mathbb{R}^N, \quad N \geq 2. \tag{E$_1$}$$

A radially symmetric solution of (E$_1$) satisfies the ordinary differential equation

$$\left(|u'|^{m-2}u'\right)' + \frac{N-1}{r}|u'|^{m-2}u' + g(r,u) = 0, \qquad R_0 < r < R_1. \tag{E$_2$}$$

With the change of variables, $t = r^{(m-N)/(m-1)}$ (for $m \neq N$) or $t = \log r$ (for $m = N$), equation (E$_2$) can be reduced to an equation of the type (E) or

$$(m-1)|u'|^{m-2}u'' + f(t,u) = 0, \qquad 0<t<1, \quad m \geq 2. \tag{E*}$$

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Conditions for the uniqueness of the general Sturm-Liouville boundary value problem

$$\left.\begin{array}{ll} \left(|u'|^{m-2}u'\right)' + f(t,u) = 0, \quad \text{in } (\theta_1,\theta_2), \quad m \geq 2 & (\mathrm{E}^*) \\ \left.\begin{array}{r} \alpha_1 u(\theta_1) - \beta_1 u'(\theta_1) = 0 \\ \alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0 \end{array}\right\} & (\mathrm{BC}^*) \end{array}\right\} \quad (\mathrm{BVP}^*)$$

have been studied by many authors; see, for example, [1–5] and the excellent book by Agarwal and Lakshmikantham [6], for $m = 2$. For the case $m \neq 2$, we refer to [7]. In this excellent paper, Naito considered the case $f(t,u) = p(t)f(u)$ and established some excellent conditions for uniqueness by using the generalized Prüfer transformation and comparison theorems.

Recently, Wong [8] used a concise approach to establish the following theorem.

THEOREM A. *(Uniqueness Theorem, see [8].) Suppose that $m \geq 2$, $\alpha_i, \beta_i \geq 0$ satisfying ${\alpha_i}^2 + {\beta_i}^2 \neq 0$ for $i = 1, 2$, and $f : [0,1) \times (0,\infty) \to (0,\infty)$ satisfying $f(t,u)$ is locally Lipschitz continuous for $u \in (0,\infty)$, and $f(t,u)/u^{m-1}$ is strictly decreasing in $u > 0$ for each fixed $t \in (0,1)$. Then (BVP*) has at most one positive solution in $C^1([\theta_1,\theta_2])$.*

Conditions for the existence of positive solutions of equation (E) with respect to (BC)–(BC*) were studied by many authors: see, for instance, [9–25] and the references therein.

The main techniques of the above results are "Fixed Point Theory", "Topological Transversality", and the "Comparison Theorem". In this paper, we apply a different approach from those used before. This method is due to Usami [26]; they considered the following boundary value problem:

$$\begin{aligned} &\Delta u + f(t,u(t)) = 0, \qquad t \in \mathbb{B}, \\ &u \in \partial\mathbb{B}, \end{aligned}$$

where $f(t,u)$ is a positive smooth function on $\mathbb{B} \times (0,\infty)$ which is nonincreasing in $u > 0$ and $\mathbb{B}$ is a unit ball. Applying the shooting method, they obtained some excellent existence results. Inspired by the power of this elementary method, we intend to construct some existence result of the boundary value problem (BVP) by using it.

2. MAIN RESULT

First, we consider the initial value problem

$$\left.\begin{array}{rll} (|u'(t)|^{m-2}u'(t))' + f(t,u(t)) = 0, & 0 < t < 1, & (\mathrm{E}) \\ u(0) = \alpha > 0, & u'(0) = 0 & (\mathrm{IC}_\alpha) \end{array}\right\} \quad (\mathrm{IVP}_\alpha)$$

and define

$$\mathbb{Z}_\alpha := \{u \mid u \text{ is a positive solution of } (\mathrm{IVP}_\alpha) \text{ in some suitable } (0,T_\alpha) \subseteq (0,\infty)\},$$

i.e., $u \in C^2[0,T_\alpha) \cap C[0,T_\alpha]$, $u(t) > 0$ on $[0,T_\alpha)$ and satisfies (IVP) in a subinterval of $[0,T_\alpha] \cap [0,1]$.

Here, we shall point out there are *few* functions that guarantee the uniqueness of positive solutions of (E) with respect to a "*zero initial condition*": $u(\eta)u'(\eta) = 0$. For example, consider

$$(|u'|^2u')' + 24(1-u) = 0; \qquad (\mathrm{IC}^{**}) \quad u(0) = 1, \quad u'(1) = 0,$$

the forcing term is infinitely smooth. Nevertheless, there are at least three solutions:

$$u_1(r) := 1, \qquad u_2(r) := 1 - r^2, \qquad \text{and} \qquad u_3(r) := 1 + r^3.$$

Therefore, $\mathbb{Z}_\alpha$ maybe has more than one element.

NOTE. The hypothesis "$f(t,x)$ is *locally Lipschitz continuous*", which guarantees the uniqueness of a positive solution of (E) with respect to *nonzero initial condition*: $u(\eta)u'(\eta) \neq 0$.

LEMMA B. *Suppose that $u_\beta(t) \in \mathbb{Z}_\beta$ is positive on $[0,T) \subseteq [0,1)$, $\alpha \geq \beta > 0$, and $u_\alpha(t) \in \mathbb{Z}_\alpha$. Then, $u_\alpha(t)$ is also positive on $[0,T)$ and satisfies*

$$u_\alpha(t) > u_\beta(t), \qquad \text{for } t \in (0,T). \tag{2.1}$$

PROOF. By Theorem A, we obtain the desired results immediately.

EXISTENCE THEOREM. *Suppose that*

$$\sup_{t\in[0,1)} \lim_{\alpha\to\infty} \frac{f(t,\alpha)}{\alpha^{m-1}} = 0, \tag{H$_1$}$$

$$\inf_{t\in[0,1)} \lim_{\alpha\to 0^+} \frac{f(t,\alpha)}{\alpha^{m-1}} = \infty. \tag{H$_2$}$$

Then the boundary value problem (BVP) has a positive solution in $C^2[0,1) \cap C[0,1]$.

PROOF. Define the subsets $\overline{S}$, $\underline{S} \subset (0,\infty)$, respectively, by

$$\overline{S} = \{\alpha > 0 : u_\alpha(t) := \inf \mathbb{Z}_\alpha \text{ exists on } [0,1) \text{ and satisfies } u_\alpha(1) > 0\},$$
$$\underline{S} = \{\beta > 0 : u_\beta(t) := \sup \mathbb{Z}_\beta \text{ vanishes before } t = 1\}.$$

It follows from Lemma B that $\alpha \in \overline{S}$ and $\beta \in \underline{S}$ implies $\alpha > \beta$.

Now, we separate the rest of the proof into the following steps.

STEP (i). *$\overline{S}$ is not empty.* We will show that $u_\alpha(t) := \inf \mathbb{Z}_\alpha > \alpha/2$ if α is sufficiently large, and therefore such an α belongs to $\overline{S}$. Following from (H$_1$), we see that there exists an $\alpha > 0$ sufficiently large such that

$$\int_0^1 \left[\int_0^s \frac{f(r,\alpha/2)}{(\alpha/2)^{m-1}}\,dr\right]^{1/(m-1)} ds < \frac{1}{2}. \tag{2.2}$$

We claim that for α satisfying (2.2), $u_\alpha(t) > \alpha/2$ for $t \in [0,1)$. In fact, if this is not the case, then there exists a $t_1 \in (0,1)$ such that

$$u_\alpha(t) > \frac{\alpha}{2}, \quad 0 \leq t < t_1; \qquad u_\alpha(t_1) = \frac{\alpha}{2}.$$

Integrating (E) and using (IC$_\alpha$), we have

$$u_\alpha(t_1) - \alpha + \int_0^{t_1} \left[\int_0^s f(r,u_\alpha(r))\,dr\right]^{1/(m-1)} ds = 0,$$

which, in view of decreasing property of $f(t,u)/u^{m-1}$, implies that

$$\begin{aligned}
\frac{\alpha}{2} &= \int_0^{t_1} \left[\int_0^s f(r,u_\alpha(r))\,dr\right]^{1/(m-1)} ds \\
&= \int_0^{t_1} \left[\int_0^s \frac{f(r,u_\alpha(r))}{(u_\alpha(r))^{m-1}} (u_\alpha(r))^{m-1}\,dr\right]^{1/(m-1)} ds \\
&< \int_0^{t_1} \left[\int_0^s \frac{f(r,\alpha/2)}{(\alpha/2)^{m-1}} (u_\alpha(r))^{m-1}\,dr\right]^{1/(m-1)} ds \\
&< \alpha \int_0^{t_1} \left[\int_0^s \frac{f(r,\alpha/2)}{(\alpha/2)^{m-1}}\,dr\right]^{1/(m-1)} ds < \frac{\alpha}{2}.
\end{aligned}$$

This contradiction shows that $u_\alpha(t) > \alpha/2$ on $[0,1)$.

STEP (ii). $\underline{S}$ *is not empty.* We claim that there exists a sufficiently small $\alpha > 0$, $u_\alpha(t) := \sup \mathbb{Z}_\alpha$ that satisfies $u_\alpha(1/2) \le 0$. Suppose to the contrary that for all $\alpha > 0$, $u_\alpha(t) > 0$ on $[0, 1/2]$. It follows from (H_2) that we can choose an $\alpha > 0$ so small such that

$$\frac{f(t,\alpha)}{\alpha^{m-1}} > \theta^{m-1}, \tag{2.3}$$

where $\theta := [\int_0^{1/2}(\int_0^s (1-2r)^{m-1}\,dr)^{1/(m-1)}\,ds]^{-1} > 0$.

Let $h(t) := -2\alpha t + \alpha$ and $U(t) := u_\alpha(t) - h(t)$ on $[0, 1/2]$. It follows from $u_\alpha(0) = \alpha = h(0)$ and $h'(0) = -2\alpha < 0 = u'_\alpha(0)$, that

$$\begin{aligned} U'(t) &= u'_\alpha(t) + 2\alpha, \\ U''(t) &= u''_\alpha(t) < 0, \qquad \text{on } \left[0, \frac{1}{2}\right] \end{aligned}$$

and there exists a $t_0 \in [0, 1/2]$ satisfying

$$U(t) \ge 0, \qquad \text{on } [0, t_0].$$

Now, we claim that $U(t) \ge 0$ on $[0, 1/2)$. Assume to the contrary that there exists a $t_1 \in [t_0, 1/2)$ such that

$$U(t_1) = 0 \quad \text{and} \quad U'(t_1) \le 0.$$

Since

$$U\left(\frac{1}{2}\right) = u_\alpha\left(\frac{1}{2}\right) - h\left(\frac{1}{2}\right) = u_\alpha\left(\frac{1}{2}\right) > 0,$$

there is a $t_2 \in (t_1, 1/2)$ such that $U'(t_2) \ge 0$, which contradicts to $U''(t) < 0$ on $[0, 1/2]$. Therefore, we obtain

$$u_\alpha(t) \ge h(t) = -2\alpha t + \alpha, \qquad \text{on } \left[0, \frac{1}{2}\right). \tag{2.4}$$

Integrating (E) and using (2.3), (2.4), and the monotone property of $f(t,u)/u^{m-1}$, we see that

$$\begin{aligned} -u_\alpha\left(\frac{1}{2}\right) &= -\alpha + \int_0^{1/2} \left[\int_0^s f(r, u_\alpha(r))\,dr\right]^{1/(m-1)} ds \\ &= -\alpha + \int_0^{1/2} \left[\int_0^s \frac{f(r, u_\alpha(r))}{(u_\alpha(r))^{m-1}} (u_\alpha(r))^{m-1}\,dr\right]^{1/(m-1)} ds \\ &> -\alpha + \int_0^{1/2} \left[\int_0^s \frac{f(r,\alpha)}{\alpha^{m-1}} (u_\alpha(r))^{m-1}\,dr\right]^{1/(m-1)} ds \\ &\ge -\alpha + \int_0^{1/2} \left[\int_0^s \frac{f(r,\alpha)}{\alpha^{m-1}} (-2\alpha r + \alpha)^{m-1}\,dr\right]^{1/(m-1)} ds \\ &= -\alpha + \alpha \int_0^{1/2} \left[\int_0^s \frac{f(r,\alpha)}{\alpha^{m-1}} (1-2r)^{m-1}\,dr\right]^{1/(m-1)} ds \\ &> -\alpha + \alpha\theta \int_0^{1/2} \left[\int_0^s (1-2r)^{m-1}\,dr\right]^{1/(m-1)} ds \\ &= -\alpha + \alpha = 0, \end{aligned}$$

implying that $u_\alpha(1/2) < 0$, a contradiction. Thus, $\underline{S}$ contains α satisfying (2.3).

STEP (iii). $\inf \overline{S}$ *does not belong to* $\overline{S}$. Let $\alpha_* = \inf \overline{S}$. It is clear that $\alpha_* \in (0, \infty)$. Suppose to the contrary that $\alpha_* \in \overline{S}$ and let $u_*(t) := \inf \mathbb{Z}_{\alpha_*}$. Then $l := u_{\alpha_*}(1) > 0$. Choose $t_1 \in (0,1)$

sufficiently close to 1 so that

$$\int_{t_1}^{1}\left[\int_0^s f\left(r,\frac{l}{2}\right)dr\right]^{1/(m-1)} ds < \frac{l^2}{4\alpha_*}. \tag{2.5}$$

Let $\beta \in (0,\alpha_*)$ sufficiently close to α_* and $u_\beta(t) := \inf \mathbb{Z}_\beta$ be defined on $[0,T_\beta]$. Since $\beta \in (0,\alpha_*)$ is near α_* and $u_{\alpha_*}(1) = l > 0$, there exists a $t_1 \in (0,T_\beta)$ such that $u_\beta(t_1) > l$. Now, we claim that such a $u_\beta(t)$ satisfies $u_\beta(t) > l/2$ on its interval of existence, and consequently can be extended to $[0,1)$. In fact, if this is not true, then there exists a $t_2 \in (t_1,1)$ such that

$$u_\beta(t) > \frac{l}{2}, \quad t_1 \le t \le t_2; \qquad u_\beta(t_2) = \frac{l}{2}.$$

Integrating (E) and our choice of t_2 yields the following condition:

$$\begin{aligned}
\frac{l}{2} = u_\beta(t_2) &= u_\beta(t_1) - \int_{t_1}^{t_2}\left[\int_0^s f(r,u_\beta(r))\,dr\right]^{1/(m-1)} ds \\
&= u_\beta(t_1) - \int_{t_1}^{t_2}\left[\int_0^s \frac{f(r,u_\beta(r))}{(u_\beta(r))^{m-1}}(u_\beta(r))^{m-1}\,dr\right]^{1/(m-1)} ds \\
&\ge u_\beta(t_1) - \int_{t_1}^{t_2}\left[\int_0^s \frac{f(r,l/2)}{(l/2)^{m-1}}(u_\beta(r))^{m-1}\,dr\right]^{1/(m-1)} ds \\
&\ge u_\beta(t_1) - \int_{t_1}^{t_2}\left[\int_0^s \frac{f(r,l/2)}{(l/2)^{m-1}}\beta^{m-1}\,dr\right]^{1/(m-1)} ds \\
&\ge u_\beta(t_1) - \int_{t_1}^{t_2}\left[\int_0^s \frac{f(r,l/2)}{(l/2)^{m-1}}(\alpha_*)^{m-1}\,dr\right]^{1/(m-1)} ds \\
&> l - \alpha_*\int_{t_1}^{1}\left[\int_0^s \frac{f(r,l/2)}{(l/2)^{m-1}}\,dr\right]^{1/(m-1)} ds \\
&= l - \frac{2\alpha_*}{l}\int_{t_1}^{1}\left[\int_0^s f\left(r,\frac{l}{2}\right)dr\right]^{1/(m-1)} ds \\
&> l - \frac{2\alpha_*}{l}\cdot\frac{l^2}{4\alpha_*} = \frac{l}{2}.
\end{aligned}$$

This is a contradiction. Therefore, such a β must be a member of $\overline{S}$. This contradicts the definition $\alpha_* = \inf \overline{S}$.

STEP (iv). $\sup \underline{S}$ *does not belong to* $\underline{S}$. Suppose that $\alpha^* = \sup \underline{S} \in \underline{S}$ and let t_1 be the point in $(0,1)$ where $u_{\alpha^*}(t)$ vanishes. Choose $T \in (t_1,1)$ arbitrarily and let it be fixed. There exists a constant $\epsilon > 0$ sufficiently small such that

$$\frac{f(t,\epsilon)}{\epsilon^{m-1}} > \eta^{m-1}, \tag{2.6}$$

where $\eta := [\int_{t_1}^T(\int_{t_1}^t(1-s/T)^{m-1}\,ds)^{1/(m-1)}\,dt]^{-1} > 0$. Let $\beta > \alpha^*$ sufficiently close to α^* and $u_\beta(t) := \sup \mathbb{Z}_\beta$ be defined on $[0,T_\beta]$. It follows from $\beta > \alpha^*$ is near α^*, $u_{\alpha^*}(t_1) = 0$, and Lemma B, that $0 < u_\beta(t_1) < \epsilon$. Now, we assert that such a $u_\beta(t)$ vanishes before $t = T$. Assume to the contrary that $u_\beta(t)$ exists on $[0,T]$ and remains positive. Then we obtain $0 < u_\beta(t) \le \epsilon$, $t_1 \le t \le T$. Similar to (2.4), we have

$$u_\beta(t) \ge -\frac{\beta}{T}t + \beta, \qquad \text{on } [0,T]. \tag{2.7}$$

Integrating (E) twice and using (2.6),(2.7), we get

$$
\begin{aligned}
-u_\beta(T) &= -u_\beta(t_1) + \int_{t_1}^{T} \left[\int_0^t f(s,u_\beta(s))\,ds\right]^{1/(m-1)} dt \\
&\geq -u_\beta(t_1) + \int_{t_1}^{T} \left[\int_{t_1}^t f(s,u_\beta(s))\,ds\right]^{1/(m-1)} dt \\
&= -u_\beta(t_1) + \int_{t_1}^{T} \left[\int_{t_1}^t \frac{f(s,u_\beta(s))}{(u_\beta(s))^{m-1}}(u_\beta(s))^{m-1}\,ds\right]^{1/(m-1)} dt \\
&> -\beta + \int_{t_1}^{T} \left[\int_{t_1}^t \frac{f(s,\epsilon)}{\epsilon^{m-1}}(u_\beta(s))^{m-1}\,ds\right]^{1/(m-1)} dt \\
&> -\beta + \int_{t_1}^{T} \left[\int_{t_1}^t \frac{f(s,\epsilon)}{\epsilon^{m-1}}\left(-\frac{\beta}{T}s+\beta\right)^{m-1} ds\right]^{1/(m-1)} dt \\
&> -\beta + \beta\eta \int_{t_1}^{T} \left[\int_{t_1}^t \left(1-\frac{s}{T}\right)^{m-1} ds\right]^{1/(m-1)} dt \\
&= -\beta + \beta = 0.
\end{aligned}
$$

This contradiction shows that such a β is contained in $\underline{S}$. However, this contradicts the definition of $\alpha^* = \sup \underline{S}$.

STEP (v). From the above observation, we see that the point $\alpha_0 = \inf \overline{S} = \sup \underline{S}$ belongs to neither $\overline{S}$ nor $\underline{S}$, and from the definition of $\overline{S}$ and $\underline{S}$, it is clear that there exists a $u \in \mathrm{Cl}\,(\mathbb{Z}_{\alpha_0}) :=$ the closure of $\mathbb{Z}_{\alpha_0}$ satisfying $u(1) = 0$. Therefore, we obtain the desired results.

REMARK C. It is clear that if $f(t,u)$ is (strictly) decreasing in $u \in (0,\infty)$ and

$$f(t,u) \equiv h(t)u^{-p},\ h(t)u^q,\ u^\alpha + u^{-\alpha},\ \sin(t)u^{-p} + \cos(t)u^q$$

for any given $p \in [0,\infty)$, $q \in [0, m-1)$, $\alpha \in [0, m-1]$, and $h \in C((0,1);[0,\infty))$, then $f(t,u)$ satisfies "$f(t,u)/u^{m-1}$ is strictly decreasing in u".

EXAMPLE D. It follows from our existence theorem, and we see that boundary value problems:

$$
\begin{gathered}
\left(|u'|^{m-2}u'\right)' + 2[t(1-t)]^2u^{-p} = 0 \text{ in } (0,1), \qquad p \in (1-m,\infty), \\
u'(0) = u(1) = 0,
\end{gathered} \tag{BVP.1}
$$

$$
\begin{gathered}
\left(|u'|^{m-2}u'\right)' + \frac{1}{1+t^2}u^q = 0 \text{ in } (0,1), \qquad q \in (-\infty, m-1), \\
u'(0) = u(1) = 0,
\end{gathered} \tag{BVP.2}
$$

$$
\begin{gathered}
\left(|u'|^{m-2}u'\right)' + \frac{1}{t^{\alpha+1}}(u^\alpha + u^{-\alpha}) = 0 \text{ in } (0,1), \qquad \alpha \in [0, m-1], \\
u'(0) = u(1) = 0
\end{gathered} \tag{BVP.3}
$$

have at least one positive solution in $C^2[0,1) \cap C[0,1]$.

REFERENCES

1. M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Rational. Mech. Anal.* **105**, 243–266 (1989).
2. R. Dalmasso, Uniqueness of positive and nonnegative solutions of nonlinear equations, *Funkcial. Ekvac.* **37**, 461–482 (1994).
3. R. Dalmasso, Uniqueness of positive solutions of nonlinear second-order equations, *Proc. Amer. Math. Soc.* **123**, 3417–3424 (1995).

4. H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Analysis T.M.&A.* **19**, 55–64 (1986).
5. F.H. Wong and S.L. Yu, Uniqueness of positive solutions for singular nonlinear boundary value problems, *Hiroshima Mathematical Journal* **26**, 233–243 (1996).
6. R.P. Agarwal and V. Lakshmikantham, *Uniqueness and Nonuniqueness for Ordinary Differential Equations*, World Scientific, Singapore, (1993).
7. Y. Naito, Uniqueness of positive solutions of quasilinear differential equations, *Differential Integral Equations* **8**, 1813–1822 (1995).
8. F.H. Wong, Uniqueness of positive solutions for Sturm-Liouville boundary value problems, *Proc. Amer. Math. Soc.* (to appear).
9. A. Granas, R.B. Guenther and J.W. Lee, Nonlinear boundary value problems for some classes of ordinary differential equations, *Rocky Mountain Jour. Math.* **10**, 35–58 (1980).
10. R.P. Agarwal and D. O'Regan, Singular boundary value problems for superlinear second order ordinary and delay differential equations, *J. Diff. Equs.* (to appear).
11. C. Bandle, C.V. Coffman and M. Marcus, Nonlinear elliptic problems in annular domains, *J. Differential Equations* **69**, 322–345 (1987).
12. C.V. Coffman and M. Marcus, Existence and uniqueness results for semilinear Dirichlet problems in annuli, *Arch. Rational Mech. Anal.* **108**, 293–307 (1989).
13. H. Dang and K. Schmit, Existence of positive solutions for semiliear elliptic equations in annular domain, *Diff. and Integ. Equs.* **7**, 747–758 (1994).
14. L.H. Erbe, Boundary value problems for ordinary differential equations, *Rocky Mountain J. Math.* **1**, 709–729 (1970).
15. L.H. Erbe, S. Hu and H. Wang, Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.* **184**, 640–648 (1994).
16. X. Garaizar, Existence of positive radial solutions for semiliear elliptic equations in the annulus, *J. Differential Equations* **70**, 69–72 (1987).
17. D. Guo and V. Lakshmikantham, Multiple solutions of two-point boundary value problems for ordinary differential equations in Banach spaces, *J. Math. Anal. Appl.* **129**, 211–222 (1988).
18. C.P. Gupta, A note on a second order three-point boundary value problem, *J. Math. Anal. Appl.* **186**, 277–281 (1994).
19. G. Iffland, Positive solutions of a problem of Emden-Fowler type with a free boundary, *SIAM J. Math. Anal.* **18**, 283–292 (1987).
20. H.G. Kaper, M. Knapp and M.K. Kwong, Existence theorems for second order boundary value problems, *Differential Integral Equations* **4**, 543–554 (1991).
21. M. del Pino, M. Elgueta and R. Manasevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, *J. Differential Equations* **80**, 1–13 (1989).
22. W.C. Lian, F.H. Wong and C.C. Yeh, On the existence of positive solutions of nonlinear second-order differential equations, *Proc. Amer. Math. Soc.* **124**, 1117–1126 (1996).
23. J. Santanilla, Nonnegative solutions to boundary value problems for nonlinear first and second order ordinary differential equations, *J. Math. Anal. Appl.* **126**, 397–408 (1987).
24. A. Tineo, A comparison theorem for second order ODEs and application to singular problems, *Jour. Diff. Equs.* **116**, 16–30 (1995).
25. F.H. Wong, Existence of positive solutions of singular boundary value problems, *Nonlinear Analysis T.M.&A.* **21**, 397–406 (1993).
26. H. Usami, On a singular elliptic boundary value problem in a ball, *Nonlinear Analysis T.M.&A.* **13**, 1163–1170 (1989).
27. R.P. Agarwal and J. Henderson, Superlinear and sublinear focal boundary value problems, *Applicable Analysis* (to appear).
28. R.P. Agarwal, Q. Sheng and P.J.Y. Wong, Abel-Gontscharoff boundary value problems, *Mathl. Comput. Modelling* **17** (7), 37–55 (1993).
29. R.P. Agarwal and P.J.Y. Wong, Existence of solutions for singular boundary value problems for higher order differential equations, *Rendiconti del Seminario Mathematico e Fisico di Milano* (to appear).
30. L.E. Bobisud, Steady state turbulent flow with reaction, *Rocky Mount. J. Math.* **21**, 993–1007 (1991).
31. W.C. Lian and F.H. Wong, Uniqueness of positive solutions for quasilinear singular boundary value problem, *Appl. Math. Lett.* **9** (6), 53–59 (1996).
32. M. del Pino and R. Manasevich, Multiple solutions for the p-Laplacian under global nonresonance, *Proc. Amer. Math. Soc.* **112**, 131–138 (1991).